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# **(2+1) dimensional black holes in warped product scheme**

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## **ABSTRACT**

Exploiting a multiply warped product manifold scheme, we study the interior solutions of the Banados-Teitelboim-Zanelli black holes and the exterior solutions of the de Sitter black holes in the (2+1) dimensions.

Keywords: warped products, BTZ metric, de Sitter metric

# 1 Introduction

The Banados-Teitelboim-Zanelli (BTZ) [1] black hole theory treats special solutions of (2+1) dimensional anti-de Sitter (AdS) gravity possessing all the properties of black holes. Charged BTZ black holes are also constructed as the analogous solutions in (2+1) dimensional AdS-Maxwell gravity [2]. Even though the BTZ black hole is a toy model in some respect, this BTZ black hole has triggered significant interests due to its connections with some string theories [3], its role in microscopic entropy derivations [4] and quantum corrected thermodynamics [5]. Moreover, the entropy and thermodynamic properties of de Sitter space have been investigated [6] to consider the emerging possibility that the real universe resembles de Sitter space. By embedding de Sitter space as a solution of string theory, various string dualities have been exploited to obtain a microscopic description. However, so far persistent efforts have failed unfortunately even to find a completely satisfactory de Sitter solution of string theory. Hopefully this situation will change in near future. Recently, exploiting the global embedding Minkowski space approach [7], several authors have shown that this approach could yield a unified derivation of thermodynamics for various curved manifolds in (2+1) dimensions [8, 9] and in (3+1) dimensions [8, 10].

However, all these solutions have been incompletely constructed only on the partial patches bounded by the event horizons of the black holes. On the other hand, the concept of a warped product manifold was introduced [11] to provide a class of complete Riemannian manifolds with everywhere negative curvature [12], and was developed to point out that some well-known exact solutions to Einstein field equations are pseudo-Riemannian warped products [13]. Furthermore, certain causal and completeness properties of a spacetime could be determined by the presence of a warped product structure [14], and general theory of warped products were applied to discuss the special cases of Robertson-Walker and Schwarzschild manifold. The role of warped products in the study of exact solutions to Einstein field equations is now firmly established to generate interest in other areas of geometry. Recently, the warped product scheme has been applied to higher dimensional theories such as the Randall-Sundrum model [15, 16, 17] in five dimension and the non-singular warped Kaluza-Klein embeddings [18] in five to seven dimensional gauged supergravity theories. Moreover, the warped product scheme was applied to investigate warping functions associated with con-

stant scalar curvature on globally null manifold [19]. Assuming the four dimensional spacetime to be a warped product of two surfaces, the four dimensional Einstein equations were also reduced to two dimensional ones to describe wormholes and domainwalls of curvature singularities [20].

In order to investigate physical properties inside the black hole horizons, we briefly review a multiply warped product manifold  $(M = B \times F_1 \times \dots \times F_n, g)$  which consists of the Riemannian base manifold  $(B, g_B)$  and fibers  $(F_i, g_i)$  ( $i = 1, \dots, n$ ) associated with the Lorentzian metric,

$$g = \pi_B^* g_B + \sum_{i=1}^n (f_i \circ \pi_B)^2 \pi_i^* g_i \quad (1.1)$$

where  $\pi_B, \pi_i$  are the natural projections of  $B \times F_1 \times \dots \times F_n$  onto  $B$  and  $F_i$ , respectively, and  $f_i$  are positive warping functions. For the specific case of  $(B = R, g_B = -d\mu^2)$ , the above metric is rewritten as

$$g = -d\mu^2 + \sum_{i=1}^n f_i^2 g_i, \quad (1.2)$$

to extend the warped product spaces to richer class of spaces involving multiply products. Moreover, the conditions of spacelike boundaries in the multiply warped product spacetimes [21] were also studied [22] and the curvature of the multiply warped product with  $C^0$ -warping functions was later investigated [23]. From a physical point of view, these warped product spacetimes are interesting since they include classical examples of spacetime such as the Robertson-Walker manifold and the intermediate zone of RN manifold [24, 25]. Recently, the interior Schwarzschild spacetime has been represented as a multiply warped product spacetime with warping functions [23] to yield the Ricci curvature in terms of  $f_1$  and  $f_2$  for the multiply warped products of the form  $M = R \times_{f_1} R \times_{f_2} S^2$ . Very recently, we have studied the interior RN-AdS spacetime by exploiting this multiply warped product scheme [26].

In this paper we will analyze the multiply warped product manifold associated with the charge black holes such as the BTZ and de Sitter (dS) metrics to investigate the physical properties inside the event horizons. We will exploit the multiply warped product scheme to investigate the interior solutions in (2+1) charged BTZ black holes in section 2, in (2+1) charged dS black holes in section 3 so that we can explicitly obtain the Ricci and Einstein curvatures inside the event horizons of these metrics.

## 2 BTZ black holes

### 2.1 Static BTZ case

In order to investigate a multiply warped product manifold for the static Banados-Teitelboim-Zanelli (BTZ) interior solution, we start with the three-metric inside the horizon

$$ds^2 = N^2 dt^2 - N^{-2} dr^2 + r^2 d\phi^2 \quad (2.1)$$

with the lapse function for the interior solution

$$N^2 = m - \frac{r^2}{l^2}. \quad (2.2)$$

Note that the event horizon  $r_H$  is given by  $r_H = m^{1/2}l$ . Furthermore the lapse function can be rewritten in terms of the event horizon as follows

$$N^2 = \frac{(r_H + r)(r_H - r)}{l^2} \quad (2.3)$$

which is well defined in the region  $r < r_H$ .

Now we define a new coordinate  $\mu$  as follows

$$d\mu^2 = N^{-2} dr^2, \quad (2.4)$$

which can be integrated to yield

$$\mu = \int_0^r dx \frac{l}{[(r_H + x)(r_H - x)]^{1/2}}, \quad (2.5)$$

whose analytic solution is of the form

$$\mu = l \sin^{-1} \left( \frac{r}{r_H} \right) = F(r). \quad (2.6)$$

Moreover, we have the following boundary conditions

$$\lim_{r \rightarrow r_H} F(r) = \frac{l\pi}{2}, \quad \lim_{r \rightarrow 0} F(r) = 0, \quad (2.7)$$

and  $dr/d\mu > 0$  implies  $F^{-1}$  is well-defined function.

Exploiting the above new coordinate (2.6), we rewrite the metric (2.1) as a warped products

$$ds^2 = -d\mu^2 + f_1(\mu)^2 dt^2 + f_2^2(\mu) d\phi^2 \quad (2.8)$$

where

$$\begin{aligned} f_1(\mu) &= \left( m - \frac{F^{-2}(\mu)}{l^2} \right)^{1/2}, \\ f_2(\mu) &= F^{-1}(\mu). \end{aligned} \quad (2.9)$$

After some algebra, we obtain the following nonvanishing Ricci curvature components

$$\begin{aligned} R_{\mu\mu} &= -\frac{f_1''}{f_1} - \frac{f_2''}{f_2}, \\ R_{tt} &= \frac{f_1 f_1' f_2'}{f_2} + f_1 f_1'', \\ R_{\phi\phi} &= \frac{f_1' f_2 f_2'}{f_1} + f_2 f_2''. \end{aligned} \quad (2.10)$$

Using the explicit expressions for  $f_1$  and  $f_2$  in (2.9), one can obtain identities for  $f_1$ ,  $f_1'$  and  $f_1''$  in terms of  $f_1$ ,  $f_2$  and their derivatives

$$\begin{aligned} f_1 &= f_2', \\ f_1' &= -\frac{f_2}{l^2}, \\ f_1'' &= \frac{f_1 f_1'}{f_2}, \end{aligned} \quad (2.11)$$

to yield the Ricci curvature components

$$\begin{aligned} R_{\mu\mu} &= -\frac{2f_1'}{f_2}, \\ R_{tt} &= \frac{2f_1^2 f_1'}{f_2}, \\ R_{\phi\phi} &= 2f_2 f_1', \end{aligned} \quad (2.12)$$

and the Einstein scalar curvature

$$R = -\frac{6}{l^2}, \quad (2.13)$$

in the interior of the static BTZ black hole horizon.

## 2.2 Charged BTZ case

Now we consider a multiply warped product manifold associated with the charged BTZ three-metric (2.1) inside the horizon with the charged lapse function [2]

$$N^2 = m - \frac{r^2}{l^2} + 2Q^2 \ln r. \quad (2.14)$$

Note that the event horizon  $r_H$  satisfies the equation  $0 = m - \frac{r_H^2}{l^2} + 2Q^2 \ln r_H$ , and for the range  $Ql < r < r_H$  we have the coordinate  $\mu$  in Eq. (2.4)

$$\mu = \int_{Ql}^r dx \frac{l}{(m - \frac{r^2}{l^2} + 2Q^2 \ln r)^{1/2}}, \quad (2.15)$$

so that  $dr/d\mu > 0$  implies  $F^{-1}$  is well-defined function.

Exploiting the above coordinate (2.15), we can obtain the warped products (2.8) with the modified  $f_1$  and  $f_2$  as below

$$\begin{aligned} f_1(\mu) &= \left( m - \frac{F^{-2}(\mu)}{l^2} + 2Q^2 \ln F^{-1}(\mu) \right)^{1/2}, \\ f_2(\mu) &= F^{-1}(\mu), \end{aligned} \quad (2.16)$$

to yield the Ricci curvature components

$$\begin{aligned} R_{\mu\mu} &= -\frac{2f_1'}{f_2} + \frac{2Q^2}{f_2^2}, \\ R_{tt} &= \frac{2f_1^2 f_1'}{f_2} - \frac{2Q^2 f_1^2}{f_2^2}, \\ R_{\phi\phi} &= 2f_2 f_1', \end{aligned} \quad (2.17)$$

and the Einstein scalar curvature

$$R = -\frac{6}{l^2} + \frac{2Q^2}{f_2^2}, \quad (2.18)$$

in the interior of the charged BTZ black hole horizon.

Now it seems appropriate to comment on the relations between the interior and exterior solutions in the charged BTZ black hole. In the exterior of the event horizon  $r_H$  where the three-metric is given by

$$ds^2 = -\left(-m + \frac{r^2}{l^2} - 2Q^2 \ln r\right)^2 dt^2 + \left(-m + \frac{r^2}{l^2} - 2Q^2 \ln r\right)^{-2} dr^2 + r^2 d\phi^2, \quad (2.19)$$

one can obtain the Ricci curvature components in terms of the warping functions  $f_1$  and  $f_2$  as follows

$$\begin{aligned} R_{rr} &= -\frac{2f_1'}{f_1^2 f_2} + \frac{2Q^2}{f_1^2 f_2^2}, \\ R_{tt} &= \frac{2f_1^2 f_1'}{f_2} - \frac{2Q^2 f_1^2}{f_2^2}, \\ R_{\phi\phi} &= 2f_2 f_1', \end{aligned} \tag{2.20}$$

and the Einstein scalar curvature identical to the interior case (2.18). Here one notes that the Ricci components  $R_{tt}$  and  $R_{\phi\phi}$  are the same as those of interior case. Moreover from the definition of the coordinate  $\mu$  in Eq. (2.4) one can obtain the identity

$$R_{\mu\mu} = f_1^2 R_{rr} \tag{2.21}$$

which is also attainable from the Ricci components  $R_{\mu\mu}$  and  $R_{rr}$  in Eqs. (2.17) and (2.20). One can thus show that all the Ricci components and the Einstein scalar curvature are identical both in the exterior and interior of the event horizon  $r_H$  without discontinuities.

### 2.3 Rotating BTZ case

Now we consider a multiply warped product manifold associated with the rotating BTZ black hole inside the horizon whose three-metric is given by

$$ds^2 = N^2 dt^2 - N^{-2} dr^2 + r^2 (d\phi + N^\phi dt)^2 \tag{2.22}$$

where the lapse and shift functions are given by

$$\begin{aligned} N^2 &= m - \frac{r^2}{l^2} - \frac{J^2}{4r^2}, \\ N^\phi &= -\frac{J}{2r^2}, \end{aligned} \tag{2.23}$$

with an angular momentum  $J$ . Note that the event horizon  $r_\pm$  satisfies the equation  $0 = m - \frac{r_\pm^2}{l^2} - \frac{J^2}{4r_\pm^2}$  to yield the lapse function in terms of the event horizons as follows

$$N^2 = \frac{(r_+^2 - r^2)(r^2 - r_-^2)}{r^2 l^2} \tag{2.24}$$

which, for the interior solution, is well defined in the region  $r_- < r < r_+$ .

Defining a new coordinate  $\mu$  as in Eq. (2.4), we obtain

$$\mu = \int_{r_-}^r dx \frac{l}{(m - \frac{r^2}{l^2} - \frac{J^2}{4r^2})^{1/2}}, \quad (2.25)$$

whose analytic solution is of the form

$$\mu = l \sin^{-1} \left( \frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} = F(r). \quad (2.26)$$

Moreover, we have the following boundary conditions

$$\lim_{r \rightarrow r_+} F(r) = \frac{l\pi}{2}, \quad \lim_{r \rightarrow r_-} F(r) = 0. \quad (2.27)$$

Note that  $dr/d\mu > 0$  implies  $F^{-1}$  is well-defined function and in the vanishing angular momentum limit  $J \rightarrow 0$ , the above solution (2.26) reduces to the static BTZ case (2.6).

Exploiting the above new coordinate (2.26), we can obtain

$$ds^2 = -d\mu^2 + f_1(\mu)^2 dt^2 + f_2(\mu)(d\phi + N^\phi dt)^2 \quad (2.28)$$

to yield the metric of the warped product form (2.8) in a comoving coordinates where one can replace<sup>1</sup>  $d\phi + N^\phi dt \rightarrow d\phi$  to obtain the modified  $f_1$  and  $f_2$  as below

$$\begin{aligned} f_1(\mu) &= \left( m - \frac{F^{-2}(\mu)}{l^2} - \frac{J^2}{4F^{-2}(\mu)} \right)^{1/2}, \\ f_2(\mu) &= F^{-1}(\mu), \end{aligned} \quad (2.29)$$

and the Ricci curvature components

$$\begin{aligned} R_{\mu\mu} &= -\frac{2f_1'}{f_2} + \frac{J^2}{f_2^4}, \\ R_{tt} &= \frac{2f_1^2 f_1'}{f_2} - \frac{J^2 f_1^2}{f_2^4}, \\ R_{\phi\phi} &= 2f_2 f_1'. \end{aligned} \quad (2.30)$$

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<sup>1</sup>Here one notes that the detector locates in the comoving coordinates with the angular velocity  $d\phi/dt = -g_{t\phi}/g_{\phi\phi} = -N^\phi$ .



Here one notes that there does not exist an additional term associated with the angular momentum  $J$  in the  $R_{\phi\phi}$  component since we have used the comoving coordinates. The Einstein scalar curvature is then given by

$$R = -\frac{6}{l^2} - \frac{J^2}{2f_2^4}, \quad (2.31)$$

in the interior of the charged BTZ black hole horizons. Note that in the  $J \rightarrow 0$  limit, the above Ricci components (2.30) and Einstein scalar curvature (2.31) reduce to the corresponding ones in the static BTZ case.

### 3 dS black holes

#### 3.1 Static dS case

In order to investigate a multiply warped product manifold for the static de Sitter (dS) exterior solution, we start with the three-metric (2.1) outside the horizon with the lapse function for the exterior solution

$$N^2 = -m + \frac{r^2}{l^2}. \quad (3.1)$$

Note the event horizon  $r_H$  is given by  $r_H = m^{1/2}l$ . Furthermore the lapse function can be rewritten in terms of the event horizon as follows

$$N^2 = \frac{(r + r_H)(r - r_H)}{l^2} \quad (3.2)$$

which is well defined in the region  $r > r_H$ .

With a new coordinate  $\mu$  as in the BTZ case we obtain

$$\mu = l \cosh^{-1} \left( \frac{r}{r_H} \right) = F(r), \quad (3.3)$$

and the boundary condition

$$\lim_{r \rightarrow r_H} F(r) = 0. \quad (3.4)$$

Note that  $dr/d\mu > 0$  implies  $F^{-1}$  is well-defined function.

Exploiting the above new coordinate (3.3), we rewrite the metric (2.1) with the lapse function (3.1) as a warped products (2.8) where

$$\begin{aligned} f_1(\mu) &= \left( -m + \frac{F^{-2}(\mu)}{l^2} \right)^{1/2}, \\ f_2(\mu) &= F^{-1}(\mu), \end{aligned} \quad (3.5)$$

to yield, in the exterior of the static dS black hole horizon, the same form of Ricci curvature components (2.12) as those of the static BTZ case, and the Einstein scalar curvature

$$R = \frac{6}{l^2}, \quad (3.6)$$

which has the opposite sign of the static BTZ result (2.13).

### 3.2 Charged dS case

Now we consider a multiply warped product manifold associated with the charged dS three-metric (2.1) outside the horizon with the charged lapse function

$$N^2 = -m + \frac{r^2}{l^2} + 2Q^2 \ln r. \quad (3.7)$$

The event horizon  $r_H$  then satisfies the equation  $0 = -m + \frac{r_H^2}{l^2} + 2Q^2 \ln r_H$ , and for the range  $r > r_H$  the coordinate  $\mu$  in Eq. (2.4) is given by

$$\mu = \int_{r_H}^r dx \frac{l}{(-m + \frac{r^2}{l^2} + 2Q^2 \ln r)^{1/2}}. \quad (3.8)$$

Note that  $dr/d\mu > 0$  implies  $F^{-1}$  is well-defined function.

Exploiting the above coordinate (2.15), we can obtain the warped products (2.8) with the modified  $f_1$  and  $f_2$  as below

$$\begin{aligned} f_1(\mu) &= \left( -m + \frac{F^{-2}(\mu)}{l^2} + 2Q^2 \ln F^{-1}(\mu) \right)^{1/2}, \\ f_2(\mu) &= F^{-1}(\mu), \end{aligned} \quad (3.9)$$

to yield, in the exterior of the charged dS black hole horizon, the same form of Ricci curvature components (2.17) as those of the charged BTZ case, and

the Einstein scalar curvature

$$R = \frac{6}{l^2} + \frac{2Q^2}{f_2^2}. \quad (3.10)$$

### 3.3 Rotating dS case

Now we consider a multiply warped product manifold associated with the rotating dS black hole outside the horizon where three-metric (2.22) is given by the lapse and shift functions are now given by

$$\begin{aligned} N^2 &= -m + \frac{r^2}{l^2} - \frac{J^2}{4r^2}, \\ N^\phi &= -\frac{J}{2r^2}. \end{aligned} \quad (3.11)$$

Note that the event horizon  $r_\pm$  satisfies the equation  $0 = -m + \frac{r_\pm^2}{l^2} - \frac{J^2}{4r_\pm^2}$  to yield the lapse function in terms of the event horizons as follows

$$N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2 l^2}, \quad (3.12)$$

which, for the exterior solution, is well defined in the region  $r > r_+$ .

With a new coordinate  $\mu$  as in Eq. (2.4) we obtain

$$\mu = \int_{r_+}^r dx \frac{l}{(-m + \frac{r^2}{l^2} - \frac{J^2}{4r^2})^{1/2}}, \quad (3.13)$$

whose analytic solution is of the form

$$\mu = l \cosh^{-1} \left( \frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} = F(r). \quad (3.14)$$

Moreover, we have the following boundary conditions

$$\lim_{r \rightarrow r_+} F(r) = 0. \quad (3.15)$$

Note that  $dr/d\mu > 0$  implies  $F^{-1}$  is well-defined function and in the vanishing angular momentum limit  $J \rightarrow 0$ , the above solution (3.14) reduces to the static dS case (3.3).

Exploiting the above new coordinate (3.14), we can obtain the metric (2.28) to yield the warped products (2.8) in a comoving coordinates where one can replace  $d\phi + N^\phi dt \rightarrow d\phi$  and the modified  $f_1$  and  $f_2$  are given as below

$$\begin{aligned} f_1(\mu) &= \left( -m + \frac{F^{-2}(\mu)}{l^2} - \frac{J^2}{4F^{-2}(\mu)} \right)^{1/2}, \\ f_2(\mu) &= F^{-1}(\mu), \end{aligned} \quad (3.16)$$

to yield, in the exterior of the rotating dS black hole horizon, the same Ricci curvature components (2.30) as those of the rotating BTZ case, and the Einstein scalar curvature

$$R = \frac{6}{l^2} - \frac{J^2}{2f_2^4}. \quad (3.17)$$

Note that in the  $J \rightarrow 0$  limit, the above Einstein scalar curvature (3.6) reduce to the corresponding ones in the static dS case.

## 4 Conclusions

We have studied a multiply warped product manifold associated with the BTZ (dS) black holes to evaluate the Ricci curvature components inside (outside) the black hole horizons. Moreover, we have shown that all the Ricci components and the Einstein scalar curvatures are identical both in the exterior and interior of the event horizons without discontinuities for both the BTZ and dS black holes. Through further investigation, it will be interesting to study the thermodynamics for interior (exterior) solutions of BTZ (dS) black holes.

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